System Identification

Lecture 5: Principle Dynamic Mode Analysis

Sahar Moghimi
Reminder

In the Modified Discrete Volterra Model:

The discrete impulse response functions \( \{b_j(m)\} \) of the filter bank constitute a complete basis for the functional space of the system kernels and are selected a priori as a general "coordinate system" for kernel representation.

\[
v_j(t) = \int_0^\mu b_j(\tau)x(t - \tau)d\tau
\]

\[
y(t) = k_0 + \sum_{r=1}^{Q} \sum_{j_1=1}^L \ldots \sum_{j_r=1}^L a_r(j_1, \ldots, j_r)v_{j_1}(t) \ldots v_{j_r}(t)
\]

\[
k_r(\tau_1, \ldots, \tau_r) = \sum_{j_1=1}^L \ldots \sum_{j_r=1}^L a_r(j_1, \ldots, j_r)b_{j_1}(\tau_1) \ldots b_{j_r}(\tau_r)
\]

However, this is not generally the most efficient representation of the system in terms of parsimony.
Motivation

- The pursuit of parsimony raises the issue of finding the "minimum set" of linear filters \{p_j(m)\} that yield an adequate approximation of the system output.

- PDM analysis was introduced by Marmarelis (1997) for this specific purpose. The minimum set is termed the "principal dynamic modes" of the system.

- The reduction in dimensionality of the functional space defined by the filter bank requires a criterion regarding the adequacy of the PDM model prediction for the given ensemble of input-output data.
General form of the modified discrete Volterra model

\[ y = V_R c_R + H_R w_0 \]

Kernel expansion coefficients

Output additive noise

Input-dependent matrix

We seek a parsimonious representation in terms of the size of the coefficient vector \( c_R \), so that the sum of the squared residuals of the model output prediction remains below an acceptable level.

\[ y = \Psi_s \gamma_s + \zeta_s \]

Coefficients

Residue

Input-dependent matrix
Task: to find a $[P_R \times P_S]$ matrix for transformation

$$V_R \Gamma = \Psi_s$$

$$y = \Psi_s \gamma_s + \zeta_s$$

So that the residue remains below an acceptable level.

Please note the deterministic (due to reducing the filter bank dimension)

and stochastic (due to additive noise) terms of the residue
The prediction error for the PDM model is:

\[ G_s = I - \Psi_s [\Psi_s' \Psi_s]^{-1} \Psi_s' \]

\[ = I - V_R \Gamma [\Gamma' V_R' V_R \Gamma]^{-1} \Gamma' V_R' \]  \hspace{1cm} (4.4)

- The prediction error for the PDM model is:
  - **Deterministic**
  - **Stochastic**

\[ \zeta_s = G_s V_R c_R + G_s w_0 \]  \hspace{1cm} (4.5)

- Mean of the Euclidean norm of prediction error (assuming white noise for simplification)

\[ E[\zeta_s' \zeta_s] = c_R' V_R' G_s' G_s V_R c_R + \sigma_0^2 \cdot \text{Tr}\{G_s' G_s\} \]  \hspace{1cm} (4.6)

- Task: find \( \Gamma \) so that the value of (4.6) is below a given threshold
For a second-order MDV model, the output signal can be expressed as a quadratic form:

\[ y(n) = v'(n)Cv(n) \]

where \( v(n) \) is the augmented vector of the filter-bank outputs at each discrete time \( n \):

\[ v'(n) = [1 \quad v_1(n) \quad v_2(n) \quad \cdots \quad v_L(n)] \] (4.8)

Please note that PDM only utilizes the 1\textsuperscript{st} and 2\textsuperscript{nd} order volterra kernels.
For second order Volterra model we have:

\[ y(n) = v'(n)Cv(n) \]

\[ v'(n) = [1 \ v_1(n) \ v_2(n) \ \cdots \ v_L(n)] \]  \hspace{1cm} (4.8)

A symmetric matrix

\[
C = \begin{bmatrix}
    c_0 & \frac{1}{2}c_1(1) & \frac{1}{2}c_1(2) & \cdots & \frac{1}{2}c_1(L) \\
    \frac{1}{2}c_1(1) & c_2(1, 1) & c_2(1, 2) & \cdots & c_2(1, L) \\
    \frac{1}{2}c_1(2) & c_2(2, 1) & c_2(2, 2) & \cdots & c_2(2, L) \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \frac{1}{2}c_1(L) & c_2(L, 1) & c_2(L, 2) & \cdots & c_2(L, L)
\end{bmatrix} \]  \hspace{1cm} (4.9)
We can decompose the matrix C (Eigen value decomposition)

\[ C = R'\Lambda R \quad (4.10) \]

Therefore

\[ y(n) = v'(n)R'\Lambda Rv(n) \]
\[ = u'(n)\Lambda u(n) \]
\[ = \sum_{i=0}^{L} \lambda_i u_i^2(n) \]
\[ u(n) = Rv(n) \]

The first element of each eigenvector corresponds to the constant (the first element of the vector \( v(n) \)). The other eigenvector elements define the transformed filter-bank outputs:

\[ u_i(n) = \mu_{i,0} + \sum_{j=1}^{L} \mu_{i,j} v_j(n) \quad (4.13) \]
The eigenvalues quantify the relative contribution of the components $u_i^2(n)$ to the model output.

\[ y(n) = v'(n)R'\Lambda Rv(n) \]
\[ = u'(n)\Lambda u(n) \]
\[ = \sum_{i=0}^{L} \lambda_i u_i^2(n) \]

Therefore:

- Inspection of the relative magnitude of the ordered eigenvalues (by absolute value) allows us to determine which components make significant contributions to the output $y(n)$. 
Having made the selection of \( H \) "significant" eigenvalues, the model output signal is approximated as

\[
\tilde{y}(n) = \sum_{i=0}^{H-1} \lambda_i u_i^2(n)
\]

\[
= \sum_{i=0}^{H-1} \lambda_i \left[ \mu_{i,0} + \sum_{m=0}^{M-1} p_i(m)x(n-m) \right]^2
\]

\[
= \sum_{i=0}^{H-1} \lambda_i \left\{ \mu_{i,0}^2 + \sum_{m=0}^{M-1} 2\mu_{i,0} p_i(m)x(n-m) + \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} p_i(m_1)p_i(m_2)x(n-m_1)x(n-m_2) \right\}
\]

\[
p_i(m) = \sum_{j=1}^{L} \mu_{i,j} b_j(m)
\]

with "principal dynamic mode" (PDM) of the system.
The resulting Volterra kernels of the PDM model are

\[ \tilde{k}_0 = \sum_{i=0}^{H-1} \lambda_i \mu_{i,0}^2 \]

\[ \tilde{k}_1(m) = \sum_{i=0}^{H-1} 2\lambda_i \mu_{i,0} p_i(m) \]

\[ = \sum_{i=0}^{H-1} \sum_{j=1}^{L} 2\lambda_i \mu_{i,0} \mu_{i,j} b_j(m) \]

\[ \tilde{k}_2(m_1, m_2) = \sum_{i=0}^{H-1} \lambda_i p_i(m_1) p_i(m_2) \]

\[ = \sum_{i=0}^{H-1} \sum_{j_1=1}^{L} \sum_{j_2=1}^{L} \lambda_i \mu_{i,j_1} \mu_{i,j_2} b_{j_1}(m_1)b_{j_2}(m_2) \]
Therefore

- We have a minimum set of linear dynamic filters (PDMs), followed by a multivariate static nonlinearity and a threshold function.

- The PDMs are obtained by performing eigen-decomposition of a matrix constructed using the first-order and second-order Volterra kernels of the system.

- The number of PDMs increases until we reach an acceptable threshold.

- This approach can be extended to higher order kernels by replacing eigen value decomposition with singular value decomposition of rectangular matrices.
Reference

Principal dynamic mode analysis of action potential firing in a spider mechanoreceptor

Georgios D. Mitsis · Andrew S. French · Ulli Höger · Spiros Courellis · Vasilis Z. Marmarelis

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Abstract The encoding of mechanical stimuli into action potentials in two types of spider mechanoreceptor neurons is modeled by use of the principal dynamic modes (PDM) methodology. The PDM model is equivalent to the general Wiener–Bose model and consists of a minimum set of linear dynamic filters (PDMs), followed by a multivariate static nonlinearity and a threshold function. The PDMs are obtained by performing eigen-decomposition of a matrix constructed using the first-order and second-order Volterra kernels of the system, which are estimated by means of the Laguerre expansion technique, utilizing measurements of pseudorandom mechanical stimulation (input signal) and the resulting action potentials (output signal). The static nonlinearity, which can be viewed as a measure of the probability of action potential firing as a function of the PDM outputs, can then be expressed as a high-pass characteristic, illustrating the importance of the velocity of slit displacement in the generation of action potentials at the mechanoreceptor output, while the second and third PDMs exhibit band-pass and low-pass characteristics, respectively. The corresponding three-input nonlinearity exhibits asymmetric behavior with respect to its arguments, suggesting directional dependence of the mechanoreceptor response on the mechanical stimulation and the PDM outputs, in agreement to our findings from a previous study (Ann Biomed Eng 27:391–402, 1999). Differences between the Type A and B neurons are observed in the zeroth-order Volterra kernels (related to the average firing), as well as in the magnitudes of the second and third PDMs that perform band-pass and low-pass processing of the input signal, respectively.
Mechanoreceptors are primary neurons that respond to mechanical stimuli
Fast Adaptation

These neurons are often the most sensitive cells to very small changes in the stimulus, such as the tactile force. These rapidly adapting cells return to a normal rate of pulses in less than 0.1 second. These delicate mechanoreceptors are generally found in the subcutaneous layer of the skin, where they are protected from the abuses which may occur at the surface. These receptors are used in human perception to detect surface, or very small vibrations in machines.

Slow Adaptation

These receptors are generally located near the surface of the skin, and are responsible for much of the static perceptive capabilities. For example, the sensitivity to temperature at the skin is generally of a slow-adapting type, as are many tactile sensors useful for maintaining grip on an object. The adaptation time scale for these cells can be from 10 to more than 100 seconds.
FIGURE 1. Responses of spider mechanoreceptor neurons to step and random stimulation. (Upper) Responses of type A and type B neurons to current steps of 100 ms duration and ~1 nA amplitude. (Lower) Part of the response from a type B neuron during stimulation with a pseudorandom Gaussian current (amplitude 8.5 nA rms). The voltage scale applies to all recordings.
Fig. 2 Representative data segments utilized for model estimation for Type A and Type B neurons. *Top panels* pseudorandom mechanical stimulation (in μm), *middle panels* output action potentials (raw data), *bottom panels* output action potentials (binary data)
Fig. 1 Principal Dynamic Mode (PDM) model of action potential firing in a spider mechanoreceptor. The PDM model is equivalent to the Wiener–Bose model with a minimum set of linear filters $L_1, L_2, \ldots, L_m$. The input signal (mechanical displacement) is fed into the PDMs of the system, the outputs of which are combined into a multiple-input static nonlinearity and threshold function to produce the model output.
Method

- The PDMs can be extracted from the estimates of the Volterra kernels of the system.
- The general discrete-time Volterra model for a finite-memory, $Q^{th}$ order nonlinear system is

$$y(n) = \sum_{q=0}^{Q} \left\{ \sum_{m_1} \ldots \sum_{m_q} k_q(m_1, \ldots, m_q) ight.$$ 

$$x(n - m_1) \ldots x(n - m_q) \} ,$$

- Employing stimulus-response data (in our case, mechanical displacements and the resulting action potentials) data the Laguerre expansion technique is used to estimate the expansion coefficients using least-squares fitting.
Fig. 3 First-order kernels for one Type A and one Type B neuron in the time and frequency (FFT magnitude) domains, averaged over ten data segments with a length of 5 s. Solid line mean, Dotted line SE. Note the high-pass characteristic in the frequency domain.
Fig. 5 Representative output prediction achieved by linear and nonlinear Volterra models for a 200 ms segment taken from a Type B neuron. Blue linear model prediction, green second-order model prediction, red third-order model prediction. Note the significant improvement achieved by the nonlinear models.
Method

- One way to estimate the PDMs from the first and second-order kernel estimates is to construct the matrix

\[ R = \begin{bmatrix} k_0 & \frac{1}{2}k_1^T \\ \frac{1}{2}k_1 & k_2 \end{bmatrix} \]

- By selecting the most important eigen values on the basis of their magnitude, the corresponding orthonormal eigenvectors define the PDMs of the system
Fig. 6 Representative PDMs, scaled by the square root of the corresponding eigenvalue, obtained from the binary output waveforms (see Fig. 2) for one Type A and one Type B neuron in the time and frequency domains. The first, second and third PDM exhibits high-pass, band-pass and low-pass characteristics, respectively. Note that the magnitudes of the second and third PDMs are larger for the Type B neuron.
The multivariate static nonlinearity is defined as the locus of points \((u_1, u_2, \ldots, u_K)\) that correspond to output action potentials.

Fig. 8 Scatter plot of the three PDM output values that correspond to action potentials (blue) for a Type B neuron. Note the similarity to the mapping of Fig. 7 (Type A neuron).
Static nonlinearity

- $f$ was constructed by first convolving the input with the PDMs,
- separating the space of the PDM outputs $(u_1, u_2, \ldots, u_K)$ into $K$-dimensional bins,
- counting the number of points corresponding to output action potentials and dividing this number by the total number of points within each bin.
Obtained signals are divided to serve for different tasks:

- Of the approximately 80,000 data points, segments of 5,000 points (corresponding to 150–250 output action potentials) were used to estimate the Volterra kernels and PDMs of the system.

- The remaining data points were employed to construct the static nonlinearity by mapping the PDM output values onto the output action potentials (60,000 points)

- and for model validation (15,000 points).
\[ f(u_1, u_2, \ldots, u_K) \] yields a measure of the probability of action potential firing as a function of the PDM outputs.
The PDM model performance (in terms of predicting output action potentials correctly) is assessed by constructing receiver operating characteristic (ROC) curves.
Fig. 13 Representative output action potential prediction for a Type B neuron for two and three PDM models. The three PDM nonlinear model yields better performance.